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An irreducible smooth non-admissible representation

by G.F. Helminck

Faculty of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, the Netherlands

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ABSTRACT

It is shown for the group of k-rational points of an affine algebraic group G with k a finite extension of Q_p that the topological irreducibility of unitary representations of G does not have to be equivalent to the algebraic irreducibility of the representation on the smooth vectors. We give for a specific G an example of an irreducible smooth representation, which is not admissible.

1.1. Let k be a finite extension of Q_p. We denote by G the group of k-rational points of an affine algebraic group. It is a totally disconnected locally compact group. Let (ϱ, V) be a representation of G on the complex vector space V. A vector v in V is called smooth if the map

$$g \to \varrho(g)(v), \quad g \in G,$$

is locally constant. The space of smooth vectors in V, V_∞, is stable under the action of G and we denote this representation by ϱ_∞. If $V = V_\infty$, then we call (ϱ, V) smooth. A smooth representation is called admissible if moreover the following condition holds: for each open subgroup K of G, the space of vectors $v \in V$ left fixed by $\varrho(K)$ is finite-dimensional.

We call a smooth representation irreducible if V and $\{0\}$ are the only G-modules in V and we call it pre-unitary if V carries a $\varrho(G)$-invariant scalar product.

It was shown in [Ja] for reductive G and in [D] for unipotent G that every irreducible smooth representation (ϱ, V) of G is admissible. This allows you to show that a unitary representation (ϱ, V) of G is topologically irreducible if and only if $(\varrho_\infty, V_\infty)$ is a smooth irreducible representation. We will show here that this does not hold for general G.

Take

$$G=\left\{\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}, a\in k^*, x\in k\right\}.$$

It is the semi-direct product of

$$H=\left\{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\middle| a\in k^*\right\} \quad \text{and} \quad N=\left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\middle| x\in k\right\}.$$

Then the general theory of Mackey tells you that if you take a non-trivial character τ of N and induces this representation to one of G, one obtains a unitary irreducible representation $I(\tau)$ of G. We will identify τ with a character of k. The standard realization of $I(\tau)$ is on the space of measurable functions $f: G\to\mathbb{C}$ satisfying

(i) $$f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)=\tau(x)f(g) \text{ a.e.}$$

(ii) $$\int_{N\backslash G} f(g)\overline{f(g)}d\dot{g}<\infty,$$

with $d\dot{g}$ a right G-invariant measure on $N\backslash G$. G acts on this space by right translations. Clearly those functions are determined by their restriction to H and if we identify H with k^*, then we get a realization on $L^2(k^*)$ and the action of G is given by

$$\left(\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\cdot f\right)(b)=\tau(bx)f(ba). \tag{1.2}$$

For any totally disconnected locally compact group T, we denote the space of all locally constant functions with compact carrier by $C_c^\infty(T)$. Clearly $C_c^\infty(k^*)$ is contained in $L^2(k^*)_\infty$ and G-invariant. As we will see further on, it is also an irreducible smooth representation of G but it is not admissible: assume for simplicity that τ is nontrivial on $\mathbb{O}$ the ring of integers of k, and trivial on $p=(\pi)$ the maximal ideal of $\mathbb{O}$. Denote for any subset A of k the characteristic function of A by χ_A. Consider the open subgroup K of G defined by

$$K=\left\{\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\middle| a\in\mathbb{O}^*, x\in p\right\}.$$

From (1.2) and the assumption on τ, it follows that all the $\chi_{\pi^m\mathbb{O}^*}$, $m\geq 0$, in $C_c^\infty(k^*)$ are left fixed by K. It will then also be clear that if $\{\lambda_m\}$ is a sequence with $\lambda_m>0$ for all m and $\sum_{m\geq 0}\lambda_m^2<\infty$, then $f=\sum_{m\geq 0}\lambda_m\chi_{\pi^m\mathbb{O}^*}$ is left fixed by $\pi(K)$ and is an example of an element in $L^2(k^*)_\infty$, which does not belong to $C_c^\infty(k^*)$. In particular $(I(\tau)_\infty, L^2(k^*)_\infty)$ is not irreducible.

As for the irreducibility of $C_c^\infty(k^*)$, consider some non-zero f in $C_c^\infty(k^*)$ and let M be the span of its G-translates. To show that $M=C_c^\infty(k^*)$, it is sufficient to prove that all the χ_{1+p^m}, with m sufficiently large belong to M, for the action of G includes all translations in k^*. The same argument allows you to assume that

$$f = \sum_{i \geq 0} f\chi_{\pi^i \mathbb{O}^*} = \sum_{i \geq 0} f_i, \text{ with } f_0 \neq 0.$$

Let g be in $C_c^\infty(k^*)$. Then we define for each unitary character σ of $\mathbb{O}^*$ and each k in $\mathbb{Z}$, the function $g(\sigma, k)$ in $C_c^\infty(k^*)$ by

$$g(\sigma, k)(x) = g(x) \int_{\mathbb{O}^*} \tau(\pi^{1-k} bx)\sigma(b) d^*b.$$

Here d^*b denotes a Haar measure on $\mathbb{O}^*$. From the action of G it will be clear that if $g \in M$, then $g(\sigma, k)$ also belongs to M. Consider now the integral $G(\sigma, k)$ given by

$$G(\sigma, k) = \int_{\mathbb{O}^*} \tau(\pi^{1-k} b)\sigma(b) d^*b.$$

We list here some properties of $G(\sigma, k)$. First of all, if $\sigma \equiv 1$, then

$$G(1, k) = \int_{\mathbb{O}} \tau(\pi^{1-k} b) db - \int_{p} \tau(\pi^{1-k} b) db$$

$$= \begin{cases} 0 & \text{if } k > 1 \\ -\operatorname{vol}(p) & \text{if } k = 1 \\ \operatorname{vol}(\mathbb{O}^*) & \text{if } k < 1. \end{cases}$$

If σ is non-trivial, then there is a $n \geq 1$ such that $\sigma | 1 + p^n \equiv 1$ and $\sigma | 1 + p^{n-1} \not\equiv 1$. Then we have

$$G(\sigma, k) = 0 \text{ if } k \neq n$$

$$|G(\sigma, n)|^2 = (\operatorname{vol}(\mathbb{O}^*) + \operatorname{vol}(p)) \operatorname{vol}(p^n).$$

For if $k > n$

$$G(\sigma, k) = \sum_{b \in \mathbb{O}^*/1+p^n} \sigma(b) \underbrace{\int_{p^n} \tau(\pi^{1-k} b(1+t)) dt}_{=\,0}.$$

If $n > k$

$$G(\sigma, k) = \sum_{b \in \mathbb{O}^*/1+p^{n-1}} \tau(\pi^{1-k} b) \underbrace{\int_{1+p^{n-1}} \sigma(bu) d^*u}_{=\,0}.$$

If $n=k$, then the Gaussian sum $G(\sigma, k)$ satisfies

$$G(\sigma, n)\overline{G(\sigma, n)} = \int_{\mathbb{O}^*}\int_{\mathbb{O}^*} \tau(\pi^{1-n}(b-c))\sigma(bc^{-1})d^*bd^*c$$

$$= \int_{\mathbb{O}^*}\int_{\mathbb{O}^*} \tau(\pi^{1-n}(u-1)c)\sigma(u)d^*ud^*c$$

$$= \int_{\mathbb{O}^*} \sigma(u)\{ \int_{\mathbb{O}} \tau(\pi^{1-n}(u-1)c)dc - \int_{p} \tau(\pi^{1-n}c(u-1))dc\}d^*u$$

$$= \int_{1+p^n} \sigma(u)\cdot \mathrm{vol}\,(\mathbb{O}^*)d^*u - \mathrm{vol}\,(p) \int_{1+p^{n-1}/1+p^n} \sigma(u)d^*u$$

$$= (\mathrm{vol}\,(\mathbb{O}^*) + \mathrm{vol}\,(p))\mathrm{vol}\,(p^n).$$

By considering a non-trivial character σ of $\mathbb{O}^*/1+p$ we see that the support of $f(\sigma, 1)$ is contained in $\mathbb{O}^*$ and that $f(\sigma, 1)\neq 0$. Hence we may assume that f already had its support in $\mathbb{O}^*$. Note that if $x\in\mathbb{O}^*$ then

$$f(\sigma, k)(x) = f(x)\sigma(x)^{-1}\cdot G(\sigma, k).$$

Thus we have reduce the question to the following: if V is a nonzero subspace of $C^\infty(\mathbb{O}^*)$, which is stable under multiplication by unitary characters of $\mathbb{O}^*$ and under translations in $\mathbb{O}^*$, is V then equal to $C^\infty(\mathbb{O}^*)$? The answer to this question is affirmative, since it boils down to the same question for the groups $\mathbb{O}^*/1+p^m$, $m>0$.

REFERENCES

[D] Van Dijk, G. – Smooth and admissible representations of p-adic unipotent groups, Compositio Mathematicce, vol. **37**, Fasc. 1, 77–101 (1978).

[Ja] Jacquet, H. – Sur les representations des groupes réductifs p-adiques, C.R. Acad. Sc. Paris, t. 280 Série A, 1271–1272 (1975).